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Bounds on Price Setting

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Abstract

I illustrate that the equilibria of games with strategic complementarities may be of little predictive value if the players have non-compact action sets. Motivated by this observation, I study a class of macroeconomic models in which all firms can costlessly choose any price at each date from a compact interval (indexed to last period's price level). I prove three results that are valid for *any* such compact interval. First, given *any* allocation, there is a (possibly time-dependent) specification of monetary and fiscal policy that implies that allocation is part of an equilibrium. Second, given any specification of monetary and fiscal policy in which the former is time invariant and the latter is Ricardian (in the sense of Woodford (1995)), there is a sequence of equilibria in which consumption converges to zero on a date-by-date basis. These first two results suggest that standard macroeconomic models without pricing bounds (be they sticky or flex price) provide a false degree of confidence in long-run macroeconomic stability and undue faith in the long-run irrelevance of monetary policy. The paper's final result constructs a non-Ricardian nominal framework (in which the long-run growth rate of nominal government liabilities is sufficiently high) that pins down a unique stable real outcome as an equilibrium.

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1 Introduction

For at least forty years, macroeconomists have sought to base their models on solid microeconomic foundations. Many macroeconomic frameworks now explicitly model the price-setting decisions of firms, as opposed to hiding their choices within the black box of the Walrasian auctioneer. But these models make an assumption that deviates from what is standard in modern microeconomics: the price-setting firms in the macro models are allowed to choose any positive price and so have *non-compact* action sets. This paper argues that this formulation of the firms' problem is wrong, and that this mistake has led to significant errors in the discipline's thinking about core macroeconomic questions like the long-run relevance of monetary policy and long-run macroeconomic stability.

To understand why models with non-compact action sets can be said to be wrong, consider a game in which two players simultaneously choose any element of the positive reals and their payoffs are the products of the differences between their choices and 1 (that is, $(a_1-1)(a_2-1)$). There is a unique equilibrium to this game: both players choose 1. But, from a predictive point of view, this conclusion is surely nonsensical: if presented with this situation, both players would attempt to co-ordinate on choosing as big a number as possible or on choosing as small a number a possible. The model is unable to deliver that (sensible) prediction because of a technical deficiency: the players' action sets are *non-compact* and so contain no definition of what is meant by "as big (or small) a number as possible".

This example illustrates a general point: it may be highly misleading to view the equilibria of models with strategic complementarities as having predictive relevance if the players in those models have non-compact action sets. The next section of this paper uses a two-period example to demonstrate that this same issue is present in macroeconomic models when price-setting firms can choose any positive price. In those models, if firms in the future are expected to choose prices that are "too low", then the current real interest rate will be "too high" (relative to some artificial moneyless equilibrium level). Then households' demand for consumption is "too low" and the real wage is also "too low". Price-setting firms try to expand production by engaging in a price war - that is, by cutting prices as much as possible.

But here we reach a dead-end: the term “as much as possible” (or for that matter, “price war”) is not defined given that the price-setting firms can choose any positive price. It is typical for modelers to respond to this situation by concluding, either implicitly or explicitly, that the real interest can never be “too high” (or, for similar reasons, never be “too low”). Of course, that argument is the same as arguing (wrongly!) in the two person co-ordination game described earlier that player 2 will choose 1 so as to ensure that player 1 has a well-defined best response. Just as was true in that game, the equilibria of macroeconomic models in which the price-setting firms have non-compact action sets lack predictive relevance.[1]

With this conclusion in mind, in Sections 3-4, I study the implications of a simple infinite-horizon macroeconomic model in which monopolistically competitive firms can all costlessly choose prices at each date from a compact positive interval (indexed to the prior period’s price level). The model features time and state invariant specifications of technology and preferences, so that equilibrium quantities would be time invariant in a moneyless economy (with labor, for example, being the numeraire). I obtain three main results. They are all valid for *any* specification of the compact price-setting interval.

The first result is that, given *any* allocation of resources, it is an equilibrium outcome for some (possibly time-dependent) nominal framework (interest rate rule and time path of nominal liabilities). This result is straightforward to prove: given an allocation, we need only pick a path of nominal interest rates that is consistent with that allocation and its associated inflation rate path. But it has important implications. Scientifically, the result implies that it is impossible to model the real economy accurately without having some minimal information about monetary policy. This undercuts the forty-year-long agenda of modeling the real economy while completely ignoring monetary policy. From a policy point of view, and relatedly, the result implies that even if prices are highly flexible, economies

[1] Asen Kochov pointed out to me that this argument is reminiscent of Jackson’s (1992) critique of the use of unbounded mechanisms in implementation theory. Bassetto and Phelan (2015) also criticize the use of non-compact action sets in macroeconomic models.

can't obtain desirable real outcomes without appropriate monetary policy interventions.

One response to this "extreme relevance of monetary policy" result is that its proof relies on the possibility that the central bank could follow "crazy" policies (specifically, time-dependent interest rate pegs). The second main result of the paper restricts attention to "sensible" nominal frameworks in which the central bank follows a time invariant monetary policy rule and in which fiscal policy is Ricardian[2] in the sense of Woodford (1995) (so that the intertemporal government budget constraint is satisfied for all time paths of inflation rates). I show that, for any such nominal framework, there is a sequence of equilibria in which consumption (which equals output and labor) converges datewise to zero.[3] The corresponding sequence of household utilities, as evaluated at the initial date, converges to zero (which is the households' utility level from a time path that delivers zero consumption and zero labor at all dates). This result implies that, even if governments use what might appear to be sensible nominal frameworks, the dynamic complementarities in monetary economies can give rise to deviations from macroeconomic stability that are large in terms of both quantities and welfare.

This second result covers two distinct cases. The first is that the monetary policy rule is such that the real interest rate paid by money is below the rate of time preference when inflation is at its lowest possible level.[4] In that case, given any initial level of consumption that is "too low" (relative to what would occur in a benchmark moneyless equilibrium), there is an equilibrium in which agents believe consumption falls to zero over time. By shrinking the initial level to zero, we can create a sequence of equilibria with consumption levels that converge datewise (indeed, uniformly over all dates) to zero.

The second case is that the monetary policy rule is such that the real interest rate paid by money is higher than the rate of time preference when inflation is at its lowest possible level.

[2] See also Leeper (1991)'s highly related discussion of active fiscal policy.

[3] A sequence of consumption paths $\{(C_t^k)_{t=1}^{\infty}\}_{k=1}^{\infty}$ converges datewise to zero if $lim_{k\to\infty} C_t^k = 0$ for all t.

[4] The intuition in this paragraph extends immediately to the case that the real interest rate paid by money is equal to the rate of time preference when inflation is at its lowest level. The only change is that consumption is constant over time, rather than falling over time.

In this case, a belief that inflation is near its minimal level at some future date T generates an equilibrium in which consumption grows until date T (and is then constant at the moneyless equilibrium level). By pushing that date T to infinity, we can create a sequence of equilibria with consumption levels that converge datewise to zero.[5]

The third and final result describes how to design a nominal framework that uniquely implements the constant moneyless equilibrium real outcome. The unique implementation is valid for any compact interval of firm pricing choices, as long as the central bank's inflation target is known to lie in the interior of that set. The previous result shows that the framework must be non-Ricardian. As in prior work on non-Ricardian fiscal policies by Benhabib, Schmitt-Grohe and Uribe (2002), I use a fiscal policy that targets the (long-run) growth rate of nominal liabilities.[6] Specifically, I consider any nominal framework in which:

1. the monetary policy rule is active (the implied real interest rate is a strictly increasing function of the inflation rate) when the inflation rate is above, at or *slightly* below target.

2. the growth rate of the government's nominal liabilities converges over time from below to the target nominal interest rate (expected inflation is at target and the real interest rate equals the rate of time preference).

(I don't impose any restrictions on monetary policy when the inflation rate is more than slightly below target to allow for the possibility of a lower bound on the nominal interest

[5]This result (about the arbitrarily bad equilibria that are possible when fiscal policy is Ricardian) applies for *any* interest rate rule. Consequently, it has nothing to do with the existence of a lower bound - zero or otherwise - on the nominal interest rate, and so is not related to the results of Benhabib, Schmitt-Grohe and Uribe (2001). Indeed, the range of indeterminacy exhibited in this paper is much broader than is established in those authors' work.

[6]I restrict fiscal policies to date-contingent lump-sum nominal transfers/taxes and assume that the only government liability is (interest-bearing) money. Given this restriction, whether or not a fiscal policy is Ricardian has to do with the present value of the long-run nominal liabilities of the government, when calculated using the *lowest* possible time path of nominal interest rates. If that present value is zero, then the government's infinite-horizon intertemporal budget constraint is satisfied for any sequence of inflation rates, and becomes irrelevant for price level determination. Note that if the lowest time path of nominal interest rates is highly negative, then the long-run growth rate of nominal liabilities must also be highly negative (so that nominal liabilities converge to zero extremely rapidly) in a Ricardian fiscal policy.

rate.) Given such a nominal framework, I establish that, in any equilibrium in which firm profits are uniformly bounded from below by a non-positive number, consumption is constant at the benchmark moneyless equilibrium level. Note that the last enumerated requirement means that fiscal policy is non-Ricardian, because the government's intertemporal budget constraint is not satisfied when the nominal interest rate is lower than the target nominal interest rate. This policy is being used to eliminate equilibria in which consumption is ever below its moneyless level.

The baseline model in this paper assumes that all firms are free to choose their prices at each date from a common compact interval. However, in Appendix A, I consider a version of the model in which only some firms are free to choose their prices from a common interval while the others' prices are fixed. The model is intended to capture the key distortions in both date-contingent and state-contingent pricing paradigms. I show that equilibrium consumption is bounded away from zero when the common interval of firm price choices equals the positive reals. In contrast, if the nominal framework is Ricardian, then there is a sequence of equilibria in which consumption converges to zero when the common interval is compact. Even when only a subset of firms can make price choices, the models provide misleading predictions when those firms have non-compact action sets.

The policy conclusions in this paper are quite different from those reached in the New Keynesian literature (as elucidated in Gali (2015), for example). That literature takes as given that, under any specification of monetary policy and Ricardian fiscal policy, aggregate outcomes remain close to a long-run zero inflation steady-state. Its main conclusion is that, given this presumption, there are no equilibrium deviations from steady-state under *active* monetary policy rules. In this paper, it is shown that *all* monetary policy rules, when combined with Ricardian fiscal policies, admit arbitrarily poor equilibrium outcomes. This finding about real outcomes echoes Cochrane's (2011) argument that, if fiscal policy is Ricardian, then inflation is indeterminate in equilibrium for all monetary policy rules (active or not). In order to ensure macroeconomic stability, governments must follow non-Ricardian fiscal policy regimes (assuming that such regimes are possible).[7]

Some readers of prior versions of this paper have suggested that it is best seen as a descendant of what might be termed an "old" Keynesian literature (such as Dreze (1975)). I see any such connection as being purely incidental. As described above, the paper uses reasoning from game theory to repair a (previously unnoticed) deficiency in the microfoundations of modern macroeconomic models. Along those lines, it would be natural to extend the paper's analysis by incorporating more explicit models of wage determination and asset price formation (as opposed to the Walrasian approach that I use).

I close the introduction with a final comment about the role of money in the models studied in this paper. Money has no transaction role. (It does serve as a unit of account, because firms denominate their prices in units of money.) In each period, agents' consumption spending equals their wage income and their share of firms profits. They hold money only to pay their taxes. Accordingly, the Friedman Rule is always satisfied: the risk-adjusted real rate of return on money is the same as that on any other asset. The point of this paper is that, even though the Friedman Rule is always satisfied and prices are arbitrarily close to fully flexible, money can be highly distortionary in this economy because its desirability as an asset is tied to household expectations about future inflation and output.

2 Why Compactness is Necessary

In this section, I use a two-period version of the infinite horizon model analyzed in the remainder of the paper to illustrate why it is necessary to assume that price-setting firms have compact action sets. In the example, a central bank pegs the nominal interest rate at $\bar{R}$. When firms can (costlessly) choose any positive price, there is a unique real consumption/output/labor C_1^{unbdd} and unique inflation rate Π_2^{unbdd} in period 2. But, as in the coordination game example in the introduction, this uniqueness result is an artificial byproduct of non-compactness. If households in period 1 expect period 2 inflation to be lower

[7] See also Cochrane (2017).

than Π_2^{unbdd}, the high real return to money means that their demand for consumption is low relative to C_1^{unbdd}. Firms try to boost that demand by engaging in a price war. This outcome (low output plus price war) seems like a reasonable description of what would happen if inflation expectations were in fact low. But non-compactness rules out this price war as a possible outcome by assuming that it cannot be resolved.

If we impose upper and lower bounds on firms' choice sets (and close them), the nature of equilibrium changes dramatically. Now, the price wars generated by low demand can occur in equilibrium. It follows that, given any $C_1^{bdd} \leq C_1^{unbdd}$, there is an equilibrium (with sufficiently low period 2 inflation expectations) in which consumption/output/labor equals C_1^{bdd} . Notably, this extreme form of real indeterminacy is valid for any positive lower bound on firm price-setting.

My baseline analysis in this section is for a macroeconomic model with flexible prices. However, in Appendix A, I show that the extreme real indeterminacy carries over to settings in which only some firms are allowed to choose prices while the rest have prices that are fixed. I also show that it generalizes to settings in which agents are able to store resources.

2.1 Two-Period Model Basics

There are two periods and a unit measure of agents who all live for two periods. There is also a unit measure of goods in period 1 and a single good in period 2. The agents maximize the expectation of a cardinal utility function of the form:

$$u\left(\left(\int c_1(j)^{1-1/\eta}dj\right)^{\frac{\eta}{\eta-1}}\right) - v(N_1) + u(C_2)$$

Here, $c_1(j)$ is consumption of good j in period 1, C_2 is consumption of the single good in period 2, and N_1 is labor in period 1. The utility function u satisfies typical restrictions:

$$u', -u'', v', v'' > 0$$

$$lim_{c\to 0} u'(c) = \infty$$

$$lim_{c\to\infty} u'(c) = 0$$

As in the body of the text, I restrict $\eta > 1$ (in order to ensure that monopolistically competitive firms have finite solutions to their maximization problems).

In period 2, the agents are each endowed with Y units of consumption.

In period 1, each good is produced by a monopolistically competitive firm that treats the wage as exogenous. Each firm has identical constant returns to scale technologies that transform a measure n units of time in period 1, $n \geq 0$, into n units of consumption goods. The agents have equal ownership of all firms. New firms are not allowed to enter and existing firms are not allowed to exit.

Money is an interest-bearing asset. Each person is endowed with M dollars of money in period 1. In period 2, money pays a gross nominal interest rate $\bar{R}$. In terms of fiscal policy, all agents are required to pay a lump-sum tax of $M\bar{R}$ dollars in period 2. Agents can trade money and consumption in period 2.

2.2 Equilibrium Definition

In any equilibrium, the households satisfy their intratemporal consumption-labor first order conditions:

$$u'(C_1^*)W^* = v'(C_1^*)P_1^* \tag{1}$$

where C_1^* is their first period consumption (aggregated over the various goods), W^* is the nominal wage, and P_1^* is the price of consumption in terms of dollars. (Here, I've exploited market-clearing by replacing labor with consumption.) Households also satisfy their intertemporal consumption-money first order conditions:

$$u'(C_1^*) = \beta \bar{R} u'(Y)/\Pi_2^* \tag{2}$$

where Π_2^* represents (gross) inflation from period 1 to period 2.

The typical firm seeks to maximize profits by choosing P_1 from an interval Λ, taking as given the price choices P_1^* of all other firms and the nominal wage W^*. It follows that:

$$P_1^* \in argmax_{P_1 \in \Lambda} P_1(P_1/P_1^*)^{-\eta} - W^*(P_1/P_1^*)^{-\eta} \tag{3}$$

An equilibrium in this economy is a specification of $(C_1^*, P_1^*, \Pi_2^*, W^*)$ that satisfies (1), (2), and (3). There are three equations and four unknowns, and so we know that equilibrium is indeterminate.

2.3 Non-Compact Choice Set: Equilibria

I consider two formulations of the firm problem. In the first, the firms' price constraint set $\Lambda = (0, \infty)$ and so firms can choose any positive price. In this case, the firm's pricing choice must satisfy the first order condition:

$$P_1^{unbdd}(1 - 1/\eta) = W^{unbdd}.$$

This implies in turn that:

$$u'(C_1^{unbdd})/v'(C_1^{unbdd}) = (1 - 1/\eta)^{-1}$$

and:

$$\Pi_2^{unbdd} = \beta \bar{R} u'(Y)/u'(C_1^{unbdd}).$$

It follows that, when the firms' pricing choices lie in a non-compact interval, there is a continuum of equilibria indexed by the period 1 price level. However, in all of these equilibria, the period 2 inflation rate Π_2^{unbdd}, period 1 consumption C_2^{unbdd}, and the real wage (W^{unbdd}/P_1^{unbdd}) are pinned down by the above (familiar) equations.

2.4 Non-Compact Choice Sets: Non-Equilibria

It is important to understand why other levels of Π_2 besides Π_2^{unbdd} can't be equilibria when firms can choose any positive price.

Suppose Π_2 were lower than Π_2^{unbdd}. Then, households' consumption C_2 would need to be lower than C_2^{unbdd} in order to satisfy their Euler equations (2). The real wage would have to satisfy (1) and so be lower than $(W^{unbdd}/P_1^{unbdd}) = (1 - 1/\eta)$.

So, far there is no reason why this low level of Π_2 can't be an equilibrium. But now consider the typical firm's profits as a function of its choice of P_1, treating the price level P_1^* and the nominal wage W^* as exogenous:

$$P_1(P_1/P_1^*)^{-\eta} - W^*(P_1/P_1^*)^{-\eta}.$$

The derivative of firm profits with respect to P_1, when evaluated at $P_1 = P_1^*$, equals:

$$[(1-\eta)P_1^* + \eta W^*]P_1^{*-\eta}$$

which is less than zero when $(W^*/P_1^*) < (1 - 1/\eta)$. The firm wants to cut P_1 below P_1^*.

Thus, there is a key complementarity between period 2 inflation and period 1 price-setting. Low expected inflation translates (through household consumption demand and labor supply) into low real wages. Faced with the low real wages, firms want to cut prices (to expand their scale of production). This complementarity implies that a natural outcome in this economy is that households expect low period 2 inflation and firms engage in a price war. But this natural outcome is simply not well-defined when the firms' constraint set is non-compact.

2.5 Compact Choice Sets: Equilibria

I now suppose that firms can choose any price in the set $[P_{min}, P_{max}]$, where P_{min} is positive and P_{max} is finite. As we saw earlier, the derivative of a firm's profit with respect to its own price choice P_1, when evaluated at a price level P_1^*, is:

$$P_1^{*-\eta}((1-\eta)P_1^* - \eta W^*).$$

The derivative is negative if:

$$W^*/P_1^* < (1 - 1/\eta).$$

and positive if:

$$W^*/P_1^* > (1 - 1/\eta).$$

Hence, the solution to the firm's problem is characterized by the first order condition:

$$\begin{aligned} P_1^* &= P_{min} \text{ if } W^* < (1-1/\eta)P_{min} \\ &= P_{max} \text{ if } W^* > (1-1/\eta)P_{max} \\ &= W^*(1-1/\eta)^{-1} \text{ if } W^*(1-1/\eta) \in [P_{min}, P_{max}]. \end{aligned}$$

We can use this first order condition to conclude that with compact price-setting, there is a continuum of equilibria that are indexed by period 2 inflation Π_2^{bdd}.

$$
\begin{aligned}
u'(C_1^{bdd}) &= \beta R u'(Y)/\Pi_2^{bdd} \\
w^{bdd} &= v'(C_1^{bdd})/u'(C_1^{bdd}) \\
P_1^{bdd} & \begin{cases} = P_{min} & \text{if } w^{bdd} < (1-1/\eta) \\ = P_{max} & \text{if } w^{bdd} > (1-1/\eta) \\ \in [P_{min}, P_{max}] & \text{if } w^{bdd} = (1-1/\eta) \end{cases} \\
W^{bdd} &= w^{bdd} P_1^{bdd}.
\end{aligned}
$$

Earlier, we saw that if firms can choose any positive price, there is a unique real equilibrium C_1^{unbdd} defined by:

$$u'(C_1^{unbdd})(1-1/\eta) = v'(C_1^{unbdd}).$$

In contrast, if firms are restricted to choose their prices from a compact interval $[P_{min}, P_{max}]$, then **any positive real allocation $C_1^{bdd} < C_1^{unbdd}$ is part of an equilibrium, regardless of the specification of P_{min}.**

Intuitively, this extreme form of real indeterminacy is a result of the strong intertemporal complementarities analyzed in the prior subsection. If expected inflation Π_2^{bdd} is low relative to Π_2^{unbdd}, then households demand little consumption in period 1 and the real wage is lower than $(1-1/\eta)$. The firms respond by engaging in a price war to expand their scale as much as possible. This line of reasoning is, so far, the same as in the prior subsection. The difference here is that, with firms are choosing from a compact set, the price war has a clear resolution: all firms choose P_{min}.

3 Infinite Horizon Model with Pricing Bounds

In this section, I describe an infinite horizon monetary model in which all firms can costlessly adjust prices subject to upper and lower bounds. The bounds are defined relative to the prior period's price level. Hence, they end up serving as constraints on inflation rates. I define and characterize equilibria in this economy.

3.1 Model Setup

Consider an economy with a unit measure of households who live forever. Time is discrete and the households maximize the expected value of:

$$\sum_{t=1}^{\infty} \beta^{t-1}(u(C_t) - v(N_t)), 0 < \beta < 1$$

where C_t is the consumption of a composite good in period t and N_t is labor in period t. Here, I assume that:

$$u', -u'', v', v'' > 0$$
$$lim_{c \to 0} u'(c) = \infty$$
$$lim_{c \to \infty} u'(c) = 0$$

and that the functions u, v are bounded from below by zero and from above by a finite number.

The composite good consists of a measure ν of consumption goods, indexed by j, and is defined as:

$$C_t = (\int_0^{\nu} (c(j)^{1-1/\eta} dj)^{\frac{\eta}{\eta-1}}, \eta > 1$$

Each household's consumption of each good j is bounded from below by zero.

Each consumption good j is produced by a monopolistically competitive firm. A typical firm j has a technology at each date that converts x units of labor into x units of consumption good j, for any $x \geq 0$. The households own equal shares of all firms. Firms are not allowed to exit or enter. (I'll discuss the import of these entry/exit restrictions later.)

Labor markets are competitive, and so, at each date, firms all hire workers at the same wage W_t (denominated in terms of dollars). Given that wage, firms simultaneously set prices for their consumption goods in terms of dollars. The firms' problems are identical, and so they each choose the same price P_t in equilibrium; that price is also the aggregate price level. At date t, each firm j is constrained to choose its price in the interval:

$$\pi_t^{UB} P_{t-1}^* \geq p_t(j) \geq \pi_t^{LB} P_{t-1}^*$$

Here, the bounds $(\pi^{UB}, \pi^{LB}) = (\pi_t^{UB}, \pi_t^{LB})_{t=1}^{\infty}$ are exogenously specified sequences. The firm treats them and last period's (endogenously determined) price level P_{t-1}^* parametrically. I define the gross inflation rate π_t as P_t/P_{t-1} and (without loss of generality) set $P_0^* = 1$.

Monetary policy works as follows. Each household is initially endowed with $\bar{M}_0$ dollars. Like reserves at many central banks, money is interest-bearing. Specifically, at the beginning of period $(t+1)$, a household that has M_t dollars is paid $(R_t(\pi_t) - 1)M_t$ dollars. Here, the interest rate rule $R = (R_t)_{t=1}^{\infty}$ is a sequence of exogenous (possibly time-dependent) weakly increasing continuous functions that map period t inflation into a period t gross nominal interest rate.

Finally, fiscal policy works as follows. The government's only liability is interest-bearing money. Let $[\bar{M}_t\}_{t=1}^{\infty}$ be an arbitrary sequence of positive real numbers. At each date $(t+1)$, the government levies a lump-sum tax, in dollars, equal to:

$$\tau_t(\pi_t) = (R_t(\pi_t) - 1)\bar{M}_t + (\bar{M}_t - \bar{M}_{t+1})$$

This tax ensures that the per-household level of nominal government liabilities at the end of period $(t+1)$ is equal to $\bar{M}_{t+1}$.

3.2 Equilibrium

In this subsection, I define an equilibrium in this economy. To simplify the analysis, I restrict attention to non-stochastic but possibly time-dependent equilibria.

I'll refer to an interest rate rule and fiscal policy $(R, \bar{M})$ collectively as a *nominal framework.* Given its specification, an equilibrium in this economy is a vector sequence $(C^*, N^*, M^*, P^*, W^*)$, where (C^*, N^*, M^*) represent per-household consumption, labor, and moneyholdings and (P^*, W^*) represent price levels and wages. Given this vector sequence, it's useful to define the implied inflation, taxes, and profits as:

$$\begin{aligned}
\pi_t^* &= P_t^*/P_{t-1}^* \\
\tau_t^* &= \bar{M}_t(R_t(\pi_t^*) - 1) + \bar{M}_t - \bar{M}_{t+1} \\
\Phi_t^* &= (P_t^* - W_t^*)N_t^*
\end{aligned}$$

The vector sequence satisfies the usual equilibrium conditions. First, (C^*, N^*, M^*) solve the household's optimization problem, given prices, wages, taxes, and profits that it treats as exogenous:

$$\begin{aligned}
(C^*, N^*, M^*) = argmax_{(C,N,M)} &\sum_{t=1}^{\infty} \beta^{t-1}(u(C_t) - v(N_t)) \\
s.t.\ P_t^* C_t + M_t &= M_{t-1}R_t(\pi_{t-1}^*) + W_t^* N_t - \tau_{t-1}^* + \Phi_t^* \ \forall t \geq 1, w.p.1 \\
C_t, M_t, N_t &\geq 0
\end{aligned}$$

Second, in any date, P_t^* solves firm j's pricing period t problem, given W_t^* and last period's price index (which shapes the bounds):

$$\begin{aligned}
P_t^* &= argmax_{P_t}(P_t^{1-\eta} - W_t^* P_t^{-\eta}) \\
s.t.\ \pi_t^{UB} &\leq P_t/P_{t-1}^* \leq \pi_t^{LB}
\end{aligned}$$

Finally, markets must clear in all dates:

$$C_t^* = N_t^*$$

$$M_t^* = \bar{M}_t$$

3.3 A Simple Characterization of Equilibrium

In this economy, there are three decisions that get made each period: consumption-savings, consumption-labor, and price-setting. The first decision gives rise to the familiar Euler equation that leaves households marginally indifferent between consumption and money:

$$u'(C_t^*) = \beta R_t(\pi_t^*) E_t(\frac{u'(C_{t+1}^*)}{\pi_{t+1}^*})$$

If we exploit goods-market clearing, the consumption-labor decision gives rise to a standard intratemporal first order condition:

$$u'(C_t^*)w_t^* = v'(C_t^*)$$

Here, w_t^* represents the period t real wage:

$$w_t^* \equiv W_t^*/P_t^*.$$

The household saving decision also gives rise to a transversality condition[8] that leaves households marginally indifferent to permanent increases/reductions in their moneyholdings:

$$lim_{t\to\infty}\beta^t u'(C_t^*)\bar{M}_t/P_t^* = 0$$

Finally, the price-setting decision on the part of the firm gives rise to the following con-

[8]In writing the transversality condition in this way, I'm implicitly restricting attention to equilibria in which the limit exists. See Kocherlakota (1992) for the relevant generalization.

dition:

$$P_t^* = max(\pi_t^{LB} P_{t-1}^*, min(\pi_t^{UB} P_{t-1}^*, (1-1/\eta)^{-1} W_t^*)).$$

In words, the firm follows the usual markup formula unless doing so violates the price bounds. If we divide through by P_{t-1}^*, we can rewrite this price-setting condition as:

$$\pi_t^* = max(\pi_t^{LB}, min(\pi_t^{UB}, (1-1/\eta)^{-1} w_t^* \pi_t^*)$$

By combining these conditions together, we can conclude that:

Proposition 1. *Given a nominal framework* $(R, \bar{M})$*, a consumption-inflation-real wage sequence* (C^*, π^*, w^*) *is part of an equilibrium if and only if it satisfies the following restrictions in all dates:*

$$\begin{aligned} u'(C_t^*) &= \beta R_t(\pi_t^*) u'(C_{t+1}^*)/\pi_{t+1}^* \\ w_t^* &= \frac{v'(C_t^*)}{u'(C_t^*)} \\ \pi_t^* &= max(\pi_t^{LB}, min(\pi_t^{UB}, (1-1/\eta)^{-1} w_t^* \pi_t^*) \end{aligned}$$

and the households' transversality condition is satisfied:

$$lim_{t\to\infty} \frac{\beta^t u'(C_t^*) \bar{M}_t}{\prod_{s=1}^t \pi_s^*} = 0$$

Proof. In Appendix B. □

3.4 Ricardian vs. Non-Ricardian Nominal Frameworks

In what follows, it will be important to distinguish between nominal frameworks $(R, \bar{M})$ that are *Ricardian* and those that are *non-Ricardian.* A nominal framework will be said to Ricardian if the limiting present value of the government's nominal liabilities is guaranteed to be zero for any possible sequence of inflation rates. Intuitively, this restriction means that, like a household in the standard definition of competitive equilibrium, the government's (intertemporal) budget constraint is satisfied for all possible price level sequences. Since the nominal interest rule consists of a sequence of weakly increasing functions, the following condition is both necessary and sufficient to ensure that the nominal framework is Ricardian.

$$lim_{t\to\infty}\frac{\bar{M}_t}{\prod_{s=1}^{t} R_s(\pi_s^{LB})} = 0. \tag{4}$$

Note that, for Ricardian nominal frameworks, the household's transversality condition is implied by the other equilibrium conditions in Proposition 1, because:

$$\frac{1}{R_t(\pi_t^{LB})} \geq \frac{1}{R_t(\pi_t^*)} = \frac{\beta u'(C_{t+1}^*)}{u'(C_t^*)\pi_{t+1}^*}.$$

This ensures that fiscal policy (that is, the specification of the path $\bar{M}$of nominal liabilities) plays no role in the determination of equilibrium.[9]

A non-Ricardian nominal framework is one in which the asymptotic growth rate of nominal liabilities is sufficiently high that the limit in (4) is positive. Under a non-Ricardian fiscal policy, it is impossible for the inflation rate to equal its minimal value for all dates in an equilibrium because such a sequence fails to satisfy the household's transversality condition. Intuitively, if the nominal liabilities are growing so rapidly while paying such a low nominal return, households would find it optimal to lower their moneyholdings permanently.

It remains a matter of some controversy among macroeconomists whether or not governments can, in fact, follow non-Ricardian policies (see Buiter and Sibert (2018)). As Kocherlakota and Phelan (1999) discuss, it is impossible using equilibrium observations to test whether or not a nominal framework is Ricardian. The household's transversality condition, and the government intertemporal budget constraint, have to be satisfied within an equilibrium. The question is about the limiting behavior of government nominal liabilities

[9] Obstfeld and Rogoff (1983) assume that the money supply (the time path of nominal government liabilities in their model) is constant. This fiscal policy is non-Ricardian, because the households' transversality condition is only satisfied if the long-run inflation rate is larger than the rate of time preference. They use this fiscal policy to eliminate hyperdeflations within their model.

at *unobserved inflation sequences.*

4 Results

In this section, I describe the three main results of the paper. All three are valid regardless of the specification of the pricing bounds.

To establish a familiar benchmark, suppose there were no pricing bounds. Then, the price-setting condition in Proposition 1 would become:

$$1 = w_t^*(1 - 1/\eta)^{-1},$$

If we assume that π_t^* is finite and positive, we can conclude that:

$$v'(C_t^*)/u'(C_t^*) = w_t^* = (1 - 1/\eta).$$

and that, in any equilibrium, output is always equal to Y^{real} where:

$$u'(Y^{real})(1 - 1/\eta) = v'(Y^{real}).$$

(The notation Y^{real} is meant to suggest that this level of output would be the equilibrium in a moneyless economy in which, for example, labor was the numeraire.) Once we add the bounds, equilibrium output may be higher or lower than Y^{real}. Proposition 1 implies that, if $C_t^* > (<) Y^{real}$, then $\pi_t^* = \pi_{max}(\pi_{min})$.

4.1 The Extreme Relevance of the Nominal Framework

In this subsection, I prove that any allocation is an equilibrium for some Ricardian nominal framework $(R, \bar{M})$.

Proposition 2. *Let C^* be any positive consumption sequence. Then there exists a Ricardian nominal framework $(R, \bar{M})$ such that C^* is part of an equilibrium given that framework.*

Proof. Given C^*, define π^* as follows:

$$\pi_t^* = \pi_t^{LB} \text{ if } C_t^* \leq Y^{real}$$
$$= \pi_t^{UB} \text{ if } C_t^* > Y^{real}.$$

Define R to be any interest rate rule (really, sequence of interest rate pegs) so that for all t and π:

$$R_t(\pi) = R_t^* \equiv (\beta^{-1} u'(C_t^*)/u'(C_{t+1}^*))\pi_{t+1}^*$$

and define $\bar{M}$ to be:

$$\bar{M}_t = \frac{\prod_{s=1}^{t} R_s^*}{t}.$$

Then, define:

$$w_t^* = v'(C_t^*)/u'(C_t^*)$$

It is readily verified, using Proposition 1, that (C^*, π^*, w^*) is part of an equilibrium given the nominal framework $(R, \bar{M})$. □

What happens in this proposition? Consider, by way of example, any period t in which:

$$\beta R_t u'(C_{t+1}^*)/\pi_{t+1}^* > u'(Y^{real}).$$

This inequality says that the nominal return on money is sufficiently high, given households' low expectations for future consumption and inflation, to lead households to demand less consumption than Y^{real}. Given the low demand for consumption, firms bid down their prices as much as possible. This same force would be at work in a model without pricing bounds but could not be reflected in an equilibrium. In a model with pricing bounds, it pushes π_t^* down to its lowest possible level.

Note that Proposition 2 is independent of the specification of the price bound sequences (π^{LB}, π^{UB}). Regardless of how wide the inflation bounds are, a macroeconomist has no information about real economic outcomes without having at least some information about the nominal framework. And, regardless of how wide the inflation bounds are, sufficiently poor choices of the nominal framework can lead to (arbitrarily) poor economic outcomes.

4.2 Real Indeterminacy in Ricardian Nominal Frameworks

The result in the prior subsection shows that, when firms have compact choice sets, it is only possible to ensure desirable real outcomes in equilibrium if the government has an appropriate nominal framework. This subsection asks a converse question: How bad can equilibrium outcomes be if the government is restricted to use "sensible" nominal frameworks? We shall see that the answer is very bad indeed.

In the remainder of the paper (not just this section), I restrict attention to environments and nominal frameworks that are time invariant. Specifically, I assume that the price-setting bounds are independent of time:

$$\pi_t^{LB} = \pi_{min}$$

$$\pi_t^{UB} = \pi_{max}$$

I require that interest rate rule is time-invariant, so that there is a weakly increasing function $\hat{R}$:

$$R_t(\pi) = \hat{R}(\pi)$$

for all t.

A nominal framework with a time-invariant interest rate rule is said to target an inflation rate π^{TAR} if:

$$\pi^{TAR} \in (\pi_{min}, \pi_{max})$$
$$\hat{R}(\pi^{TAR}) = \beta^{-1}\pi^{TAR}$$

The following proposition shows that inflation-targeting regimes implement the "natural" outcomes in which the real outcome is constant at Y^{real} and inflation is constant at π^{TAR}.

Proposition 3. *Suppose $(R, \bar{M})$ is a Ricardian nominal framework with a time invariant interest rate rule that targets π^{TAR}. Then, there is an equilibrium consumption-real wage-inflation sequence (C^*, w^*, π^*) such that for all t:*

$$C_t^* = Y^{real}$$
$$w_t^* = (1 - 1/\eta)$$
$$\pi_t^* = \pi^{TAR}$$

Proof. It is readily verified that (C^*, w^*, π^*) satisfy the conditions in Proposition 1. □

However, the next proposition demonstrates there can be many other equilibria associated with a Ricardian nominal framework with a time-invariant interest rate rule that targets π^{TAR}. As in the literature about the so-called neo-Fisherian determination of inflation rates[10], it considers a time-invariant interest rate peg.

Proposition 4. *Consider a Ricardian nominal framework with a time-invariant interest rate peg $\hat{R}$ such that:*

$$\hat{R}(\pi) = \bar{R} \equiv \beta^{-1}\pi^{TAR} \text{ for all } \pi.$$
$$\bar{M}_t = \frac{\bar{R}^t}{t} \text{ for all } t.$$

[10]See, among others, Garcia-Schmidt and Woodford (2015), Cochrane (2016), and Schmitt-Grohe and Uribe (2017)).

Then, for any horizon $K > 0$, there is a consumption-inflation sequence (C^{K*}, π^{K*}):

$$u'(C_t^{K*}) = \frac{(\pi^{TAR})^{K-t} u'(Y^{real})}{(\pi_{min})^{K-t-1} \pi_K^{K*}}, t < K$$

$$C_t^{K*} = Y^{real}, t \geq K$$

$$\pi_t^{K*} = \pi_{min}, t \leq (K-1)$$

$$\pi_K^{K*} \in [\pi_{min}, \pi^{TAR})$$

$$\pi_t^{K*} = \pi^{TAR}, t \geq K+1$$

that is part of an equilibrium. The sequence of utilities $W^{K*} \equiv \sum_{t=1}^{\infty} \beta^{t-1}[u(C_t^{K*}) - v(C_t^{K*})]$ converges to zero.

Proof. If we set the real wage $w^{K*} = v'(C^{K*})/u'(C^{K*})$, then it is easy to check that $(C^{K*}, w^{K*}, \pi^{K*})$ satisfy the conditions of Proposition 1. □

The proposition shows that a constant interest rate peg admits a class of equilibria in which the initial level of economic activity is below Y^{real}. *Within* any of these equilibria, the economy converges in *finite* time to the targeted real outcome Y^{real} and the targeted inflation rate $\pi^{TAR} = \beta \bar{R}$. This convergence is consistent with the neo-Fisherian logic that, under a nominal interest rate peg, the long-run inflation rate has to increase one-for-one with the level of the peg.

However, once we look *across* equilibria, we see that this result is highly misleading because convergence can take an arbitrarily long period of time. Indeed, as we drive K large, the limiting equilibria become extremely undesirable as the long-run becomes irrelevant:

$$lim_{K \to \infty} C_t^{K*} = 0$$

$$lim_{K \to \infty} \pi_t^{K*} = \pi_{min}$$

Technically, given any date and any ϵ, we can find an equilibrium in which, at that date (and all earlier ones), consumption/output is less than ϵ, and inflation is equal to its lowest possible level π_{min}.

Intuitively, these near-zero equilibria are generated by the dynamic strategic complementarities within the model. For example, in the equilibrium indexed by K, firms set their prices in period K so that inflation is less than π^{TAR}. As a result, money has a high gross real rate of return (relative to $1/\beta$) from period $(K-1)$ to period K. Given that high real return for money, agents' demand for consumption in period $(K-1)$ is lower than Y^{real}. Firms respond to that low demand by bidding inflation down to its lowest possible level (π_{min}) in period $(K-1)$. We can recurse backwards using the same logic to generate the low output-inflation outcomes in periods prior to K.

Of course, interest rate pegs are well-known to have undesirable properties relative to interest rate rules that respond aggressively to the inflation rate (Sargent and Wallace (1975)). But the following proposition shows that we can generalize Proposition 4 to *any* Ricardian nominal framework with a time-invariant interest rate rule (including those that are active).

Proposition 5. *Consider any Ricardian nominal framework* $(R, \bar{M})$ *that has a time invariant interest rate rule* $\hat{R}$ *that targets* π^{TAR}*. Then, there exists a sequence (indexed by* k*) of consumption-inflation sequences* $(C^{k*}, \pi^{k*})_{k=1}^{\infty}$ *that are parts of equilibria and such that for all* $t \geq 1$ *:*

$$C_t^{k*} \leq Y^{real}, k \geq 1$$

$$lim_{k\to\infty} C_t^{k*} = 0$$

$$lim_{k\to\infty} W^{k*} = 0$$

$$lim_{k\to\infty} \pi_t^{k*} = \pi_{min}$$

where $W^{k*} \equiv \sum_{t=1}^{\infty} \beta^{t-1}[u(C_t^{k*}) - v(C_t^{k*})]$.

Proof. In Appendix B. □

Proposition 5 shows that for a given Ricardian nominal framework, equilibrium outcomes can be arbitrarily bad in a welfare sense. Note that, like Proposition 2, it is valid for *any* specification of the lower bound π_{min} on inflation.

The proposition covers two possible scenarios. In the first, the real rate of return on money, when inflation equals π_{min}, is higher than the rate of time preference. This kind of monetary policy rule induces equilibria that resemble those described in Proposition 4. In the second scenario, the real rate of return on money, when inflation is at its lowest level π_{min}, is less than or equal to the rate of time preference. In this situation, there is a class of equilibria in which consumption starts at some initial level C_1^* below Y^{real} and then stays at or below C_1^*. This class of equilibria converges to zero datewise if we take C_1^* to zero.

4.3 Unique Implementation

The prior section emphasized that, under Ricardian nominal frameworks, the model economy described in Section 3 give rise to low output-low inflation equilibria that can be arbitrarily close to zero. In this subsection, I describe a class of nominal frameworks that, despite the dynamic complementarities, serve to implement a unique real outcome. It is independent of the upper/lower bounds on inflation (except for the knowledge that target inflation is strictly between them).

Proposition 5 tells us that any such class must be restricted to non-Ricardian nominal frameworks. As noted earlier, many economists are uncomfortable with non-Ricardian fiscal policies. For those economists, Proposition 5 is really the end of the story of what happens once firms' pricing choices are correctly restricted to lie in a compact set. But others (such as Cochrane (2011)) have argued that non-Ricardian fiscal policy is essential for price level determinacy, and the following Proposition is congruent with this thinking.[11]

Proposition 6. *Consider a nominal framework* $(R, \bar{M})$ *with a time-invariant interest rate*

[11] Propositions 6 and 7 can be readily extended to eliminate real outcomes other than Y^{real} in stochastic (sunspot) equilibria

rule $\hat{R}$ that targets π^{TAR} and such that, for some $\epsilon > 0$, the gross real interest rate:

$$\hat{R}(\pi)/\pi$$

is strictly increasing for $\pi \in [\pi^{TAR} - \epsilon, \pi^{TAR}]$. Suppose that fiscal policy $\bar{M}$ takes the following form:

$$\bar{M}_t = \frac{M_0}{t}\hat{R}(\pi^{TAR})^t$$

so that the rate of growth of nominal liabilities asymptotes from below to $(\hat{R}(\pi^{TAR}) - 1)$. Then, in any equilibrium, $C_t^* \geq Y^{real}$.

The proposition shows that is possible to eliminate the bad equilibria caused by money runs with a two-pronged approach. First, fiscal policy is non-Ricardian: nominal liabilities grow so rapidly over time so that households can improve their welfare by permanently reducing their moneyholdings unless the long-run nominal return on money is at least $\hat{R}(\pi^{TAR}) = \beta^{-1}\pi^{TAR}$. Second, monetary policy is active when inflation is slightly below target. This ensures that, if $C_t^* < Y^{real}$, inflation converges to some rate that is strictly below π^{TAR}, so that the long-run nominal return on money has to be less than $\beta^{-1}\pi^{TAR}$.

Other than in Proposition 2, I have ignored the possibility of high inflation-high output equilibria. As the following proposition demonstrates, these equilibria can be eliminated if monetary policy is active when inflation is high *and* it is known that firm profits are uniformly bounded from below (to reflect the possibility of firm exit).

Proposition 7. *Suppose that the interest rate rule $\hat{R}$ is time invariant and targets π^{TAR}. Suppose too that it satisfies:*

$$\frac{\hat{R}(\pi_{max})}{\pi_{max}} > \frac{\hat{R}(\pi^{TAR})}{\pi^{TAR}} = \beta^{-1}.$$

Consider the set of equilibria in which there exists $\Upsilon \geq 0$ such that firm profits (in terms of *goods are bounded from below by $-\Upsilon$):*

$$N_t^* - w_t^* N_t^* \geq -\Upsilon \text{ for all } t \geq 1$$

In any such equilibrium, consumption at any date satisfies $C_t^* \leq Y^{real}$.

Proof. In Appendix B. □

To summarize, consider a non-Ricardian nominal framework in which monetary policy is active above, at, or just slightly below the inflation target. Then, in any equilibrium in which firm profits are uniformly bounded from below by a fixed non-positive number, consumption equals Y^{real}.

5 Discussion

In this section, I discuss several aspects of the above analysis: the robustness of the results to adding currency, the robustness of the results to allowing for firm entry, the robustness of the results to richer models of demand, the non-robustness of "natural" (constant moneyless) equilibrium, and an explanation of the sources of multiple equilibrium.

5.1 Money and Currency

In the model described in Sections 3-4, money has no liquidity role and is not used in exchange. Money is held only to pay lump-sum taxes levied by the government and pays the same real return as all other assets in the economy. It is nonetheless potentially distorting because households can contemplate off-equilibrium trades of consumption for money.

In reality, households do hold non-interest-bearing currency, and banks can always trade their interest-bearing reserves with the government for that currency. How would adding this kind of asset to the model affect the results obtained in Section 4? Suppose in particular that households get utility from the real value of their currency holdings X according to the function:

$$U^m(X/P)$$

I assume that there exists a satiation level $\bar{m}$ such that:

$$U^m(m) = U^m(\bar{m})$$

for all $m \geq \bar{m}$.

Adding currency in this way has two effects on the economy. The first effect is that the central bank can no longer set the gross nominal interest rate below one. Banks can exchange their interest-bearing reserves with the government. Hence, if the reserves pay a negative (net) nominal interest rate, that interest rate is not relevant in equilibrium.

This observation about a lower bound on the nominal interest rate restricts the scope of Proposition 2. In particular, suppose that C^* is a consumption sequence such that $C^*_{t+1} < Y^{real}$ and consumption grows so slowly from period t to period $(t+1)$ that:

$$\frac{U'(C^*_{t+1})}{U'(C^*_t)} > \pi_t^{LB}/\beta.$$

Given such a consumption sequence, any gross nominal interest rate R^*_t that satisfies the Euler Equation must be less one. There is no nominal framework with non-interest-bearing currency such that C^* is an equilibrium.

The second effect of adding currency is that there is now a Friedman Rule argument that pins down an efficient rate of inflation. In particular, it is distortionary for real currency holdings to ever be below the satiation level $\bar{m}$. Hence, it is efficient for the gross nominal interest rate to be 1, and for the central bank to target a gross inflation inflation π^{TAR} equal to β.

This observation about efficiency, together with the above point about a lower bound on the nominal interest rate, means that in any time-invariant nominal framework, the monetary policy rule must set the nominal interest rate equal to one for all π below the efficient target of β, including π_{min}. It follows that any nominal framework that targets an efficient inflation rate is necessarily Ricardian, and Proposition 5 implies that it admits a sequence of equilibrium consumption sequences that converge datewise to zero.

Thus, when agents hold non-interest-bearing currency for liquidity purposes, it is impossible for the government to find a nominal framework that simultaneously eliminates liquidity distortions and the possibility of highly adverse real outcomes. Note though that Propositions 6 and 7 show that, for any $\epsilon > 0$, the government can find a nominal framework which targets $\pi^{TAR} = \beta + \epsilon$, has a gross nominal interest rate that is bounded below by one, and uniquely implements the constant real outcome Y^{real}. In this sense, the liquidity distortion needed to uniquely implement Y^{real} can be made arbitrarily small.

5.2 Firm Entry

Firm entry is not allowed in the model of Section 3.[12] This restriction may seem artificial, especially over long horizons. In this subsection, I discuss how the potential gains to entry behave in the various low-output equilibria discussed in Section 4.2.

Proposition 5 states that, for any Ricardian nominal framework and for any $\bar{\Delta} > 0$, we can find an equilibrium in which lifetime household utility is less than $\bar{\Delta}$. This result might suggest that the gains to possible entry (by a monopolistically competitive firm who adds a new variety of good) are correspondingly large. But this isn't true. If output is close to zero, firm profits (denominated in consumption goods) must also be close to zero. This means that any potential entrant who faces a fixed cost in consumption goods has less incentive to pay that cost.

[12]Firm exit is also not allowed. However, firm profits remain positive in all of the low-output equilibria studied in Section 4.2.

5.3 Richer Demand Structures

In this paper, I use the standard Dixit-Stiglitz model of monopolistic competition, with its fixed markups. In this subsection, I show that the basic argument in Section 2 regarding the need for price bounds generalizes to richer demand structures.

First, we can allow for the elasticity of demand to be variable.[13] Suppose that all firms retain the same one-for-one technology as above, but we alter preferences over the various consumption goods that make up the composite good so that each firm's demand function is $Q\psi(p/P)$. Here, p is the firm's own price, P is an aggregate price index, Q is a endogenously determined constant of proportionality that is exogenous to the firm, and ψ is a strictly decreasing function. With this specification of demand, the derivative of the firm's profits, with respect to its choice of p, is proportional to:

$$\psi'(p/P) + \psi(p/P) - (W/P)\psi'(p/P)$$

In an equilibrium, (p/P) must equal one (because the firms are symmetric). The analysis in sections 2-4 then generalizes to this case, if we substitute $\psi'(1)/\psi(1)$ for η. The variability in elasticity (with respect to relative price) has no effect, because all firms charge the same price.

Second (and more interestingly), suppose that, given this richer demand structure, firms differ in their productivities, so that a given firm can produce AL units of output by using L units of labor, where the parameter $A \in \Sigma$ differs across firms. I let $\mu(B)$ denote the measure of firms with productivities in the set $B \subseteq \Sigma$.

If a firm has labor productivity A, the derivative of its profits with respect to p is proportional to:

$$\psi'(p/P) + \psi(p/P) - \frac{W}{PA}\psi'(p/P).$$

[13] The model of demand considered here is a specialization of that used by Arkolakis, Costinot, Donaldson, and Rodriguez-Clare (2018). It nests Mongey (2018)'s model of inter-firm competition.

Define a function $f(x)$ to be the solution to:

$$\psi'(f(x)) + \psi(f(x)) - x\psi'(f(x)) = 0.$$

Then, $Pf(w/A)$ is the firm's chosen price if it can choose any positive price, when its productivity is A, the nominal wage is W, the price level is P, and the implied real wage is w. I assume that f is strictly increasing (so that low productivity firms charge higher prices).

Suppose there is some homogeneous of degree one function H such that the price index is defined as:

$$H((P_A)_{A\in\Sigma}) = P$$

where P_A is the price set by a firm with productivity A. Then, if firms can choose any positive price, the equilibrium real wage must satisfy the restriction:

$$H((f(w^{real}/A))_{A\in\Sigma}) = 1.$$

That real wage pins down the endogenous constant of proportionality Q, an aggregate amount of composite consumption C^{real}, and an aggregate amount of labor L^{real} through the first order condition between consumption and labor:

$$u'(C^{real})w^{real} = v'(L^{real}). \tag{5}$$

where:

$$C^{real} = Q \int_{A\in\Sigma} \psi(f(w^{real}/A))d\mu \tag{6}$$

$$L^{real} = Q \int_{A\in\Sigma} \psi(f(w^{real}/A))A^{-1}d\mu \tag{7}$$

In any equilibrium, when the firms can choose any positive price, $C_t^* = C^{real}$ in all dates.

Why can't a lower level of consumption be part of an equilibrium? The argument is the same as in Section 2. If we keep the same w^{real} (or raise it), then Q has to fall in order to satisfy (6) with the lower level of consumption. But that leads to a violation of the consumption-labor first order condition (5). We have to lower the real wage to, say, $w' < w^{real}$. The problem is that, if the price level is P and the nominal wage is $w'P$, then a firm with productivity A wants to set its price equal to $Pf(w'A^{-1})$. These choices combine to form a price level lower than the posited price level P, which means that - as in Section 2 - firms gain by engaging in a price war. If the firms have non-compact action sets, there is no way to resolve this price war in equilibrium.

As in Section 2, the situation changes once we add bounds. Suppose for example that all firms must choose their prices in period t from the compact interval $[\pi_{min}P^*_{t-1}, \pi_{max}P^*_{t-1}]$. Then, given any $0 < C^*_t < C^{real}$, there can be an equilibrium in which period t consumption equal C^*_t. In that equilibrium, the variables w^*_t, L^*_t, and Q^*_t satisfy:

$$u'(C^*_t)w^*_t = v'(L^*_t). \tag{8}$$

and:

$$C^*_t = Q^*_t \int\limits_{A \in \Sigma} \psi(f(w^*_t/A))d\mu \tag{9}$$

$$L^*_t = Q^*_t \int\limits_{A \in \Sigma} \psi(f(w^*_t/A))A^{-1}d\mu \tag{10}$$

The key is that in such an equilibrium, $w^*_t < w^{real}$, so that there are at least some (sufficiently high-productivity) firms that are constrained by the pricing lower bound.

Unlike in the homogeneous firm case, aggregate consumption being lower than C^{real} need not imply that inflation is at its lowest level π_{min} (because some sufficiently low-productivity firms remain unconstrained). But suppose that C^*_t is close enough to zero that the level of

w_t^* that solves (8)-(10) satisfies:

$$w_t^* \leq A_{min}(1 + \frac{\psi(1)}{\psi'(1)}),$$

where A_{min} is the lowest level of productivity in Σ. Then all firms are constrained by their pricing lower bound, and the inflation rate is π_{min}.

5.4 The Non-Robustness of the "Natural" Equilibria

I now return to the model described in Section 3 (with constant demand elasticity and homogeneous productivities). Suppose the interest rate rule is a time invariant function $\hat{R}$ that targets $\pi^{TAR} \in (\pi_{min}, \pi_{max})$. There is an equilibrium in which $C_t^* = Y^{real}$ and $\pi_t = \pi^{TAR}$ for all t. This equilibrium no doubt seems natural, given the time invariance of the various exogenous elements. However, when the nominal framework is Ricardian, the equilibrium is actually highly sensitive to small changes in inflation.

To see this point, it is helpful to consider two distinct cases. Suppose first that $\beta\hat{R}(\pi)/\pi$ is strictly increasing for $\pi \leq \pi^{TAR}$, so that **monetary policy is active for interest rates below target**. Consider an equilibrium in which date t inflation is slightly less than π^{TAR} and $C_t^* = Y^{real}$. Then:

$$\begin{aligned} \frac{u'(C_t^*)}{\pi_t^*} &= \frac{\beta\hat{R}(\pi_t^*)}{\pi_t^*}\frac{u'(C_{t+1}^*)}{\pi_{t+1}^*} \\ &< \frac{u'(C_{t+1}^*)}{\pi_{t+1}^*}. \end{aligned} \tag{11}$$

If $\pi_{t+1}^* \geq \pi_t^*$, then the firms' pricing first order conditions imply that $C_{t+1}^* \geq Y^{real}$, which contradicts (11). Hence, $\pi_{t+1}^* < \pi_t^*$, and by induction, it follows that:

$$\frac{u'(C_{t+s}^*)}{\pi_{t+s}^*} \geq [\frac{\beta\hat{R}(\pi_t^*)}{\pi_t^*}]^{-s}\frac{u'(C_t^*)}{\pi_t^*} \text{ for all } s \geq 1.$$

Since π^*_{t+s} is bounded from below by π_{min}, we can conclude that:

$$lim_{s\to\infty} C^*_{t+s} = 0.$$

Thus, when monetary policy is active, small perturbations in current inflation give rise to large changes in long-run equilibrium output/consumption.

Now suppose instead that $\hat{R}$ is strictly increasing and $\beta\hat{R}(\pi)/\pi$ is strictly decreasing for $\pi \leq \pi^{TAR}$, so that **monetary policy is passive for inflation rates below target.** Pick some period T and suppose π^*_T is slightly below π^{TAR} in that period. In that period, $C^*_t = Y^{real}$. We can solve for inflation backwards in time from date T using the following relationship:

$$\beta\hat{R}(\pi^*_{T-n}) = \pi^*_{T-n+1}, n \geq 1.$$

There is some N such that $\pi_{min} < \pi^*_{T-N+1} < \beta\hat{R}(\pi_{min})$; let $C^*_{T-t} = Y^{real}$ for all $t \leq (N-1)$. Then, define:

$$u'(C^*_{T-N}) = \beta\hat{R}(\pi_{min})\frac{u'(Y^{real})}{\pi^*_{T-N+1}}$$
$$u'(C^*_{T-r}) = \frac{\beta\hat{R}(\pi_{min})}{\pi_{min}}u'(C^*_{T-r+1}), r \geq (N+1)$$

By making T large and keeping ϵ fixed, we can generate equilibria in which C^*_1 is close to zero and inflation is slightly below target in period T. Thus, when monetary policy is passive, current real outcomes are highly sensitive to expectations about inflation in the distant future.

5.5 The Source of Multiple Equilibria

It is possible that at least some readers of this paper are thinking about Proposition 5 that, "Hmm ... I don't believe that firms face bounds on their pricing and I don't like models with multiple equilibria. So, let's get rid of the bounds and get rid of the multiple equilibria." The problem with this argument is that bounds aren't the true economic source of the multiple equilibria. Recall the co-ordination example in the introduction, and suppose that the players' action sets were a subset $[a_{min}, a_{max}]$ of the positive reals such that $a_{min} < 1 < a_{max}$. Then, there would be three equilibria $\{(a_{min}, a_{min}), (1, 1), (a_{max}, a_{max})\}$, instead of just one $\{(1, 1)\}$. In a formal sense, it is true the additional equilibria in this game are generated by compactifying the action set. But the complementarities pushing towards the extremal outcomes were in the game even without bounds. The bounds only serve to allow them to appear in the equilibrium set.

This same logic is true of the monetary economies studied in this paper. Regardless of the firms' action sets, households' perceptions about the real return to money depend on their expectations about inflation. And, regardless of the firms' action sets, households' shadow real interest rates depend on their expectations about future consumption. These forces are always operate to link these expectations about the future with current outcomes. The bounds only serve to make these complementarities manifest in the equilibrium set - and the relevant forces turn out to be highly powerful ones indeed.

This is not the first paper to note the (unrealistically?) strong power of the intertemporal complementarities in representative agent macroeconomic models. They are, for example, the source of the so-called forward guidance puzzle (del Negro, Giannoni, and Patterson (2015); McKay, Nakamura, and Steinsson (2017)).[14]

6 Conclusion

Go back to the second paragraph of the introduction. Do you agree that (1,1) is a nonsensical prediction for the outcome of this two-person game? If you do, you've already accepted the primary message of the paper: macroeconomic models with non-compact action sets for price-setting firms make misleading predictions because they lack valid microfoundations.

[14] Farhi and Werning (2017) and Gabaix (2018) analyze bounded rationality modifications of the standard model in which these intertemporal complementarities are dampened. There is an ongoing debate about how the magnitude of these intertemporal complementarities is affected by the introduction of incomplete financial markets - see, among others Werning (2015) and Kaplan, Moll, and Violante (2018).

This comment applies, of course, to any papers that model price-setting explicitly (including the large literature that treats state-dependent pricing via price-adjustment costs). But - just as importantly - it also applies to the enormous literature that uses the black box of the Walrasian auctioneer to enshroud the price-setting behavior of firms. The main finding in this paper is that standard macroeconomic models without pricing bounds (be they sticky or flex price) provide a false degree of confidence in long-run macroeconomic stability and undue faith in the long-run irrelevance of monetary policy.

The paper's results imply that governments can only ensure macroeconomic stability real outcomes if they follow non-Ricardian fiscal policies. This finding places new emphasis on old questions: can governments follow non-Ricardian fiscal policies? And, if they have this capability, do they exploit it in reality? Addressing these questions in a compelling fashion will require a deeper modeling and understanding of fiscal policy than is incorporated into current macroeconomic theory.[15]

[15] Bassetto (2002) represents an early effort along these lines.

References

[1] Arkolakis, C., Costinot, A., Donaldson, D., and Rodriguez-Clare, A., forthcoming, "The Elusive Pro-Competitive Effects of Trade," *Review of Economic Studies.*

[2] Bassetto, M., 2002, "A Game-Theoretic View of the Fiscal Theory of the Price Level," *Econometrica* 70, 2167-2195.

[3] Bassetto, M., and Phelan, C., 2015, "Speculative Runs on Interest Rate Pegs," *Journal of Monetary Economics* 73, 99-114.

[4] Benhabib, J., Schmitt-Grohe, S., and Uribe, M., 2001, "Perils of the Taylor Rule," *Journal of Economic Theory* 96, 40-69.

[5] Benhabib, J., Schmitt-Grohe, S., and Uribe, M., 2002, "Avoiding the Liquidity Trap," *Journal of Political Economy* 110, 535-563.

[6] Buiter, W., and Sibert, A., 2018, "The Fallacy of the Fiscal Theory of the Price Level - One Last Time," *Economics: The Open-Access, Open-Assessment E-Journal* 12 (2018-48), 1–56

[7] Cochrane, J., 2011, "Determinacy and Identification with Taylor Rules," *Journal of Political Economy* 119, 565-615.

[8] Cochrane, J., 2016, "Do Higher Interest Rates Raise or Lower Inflation?" The University of Chicago working paper.

[9] Cochrane, J., 2017, "Michelson-Morley, Occam and Fisher: The Radical Implications of Stable Quiet Inflation at the Zero Lower Bound," *NBER Macroeconomics Annual* 32, edited by Martin S. Eichenbaum and Jonathan Parker.

[10] Del Negro, M., Giannoni, M., and Patterson, C., 2015, "The Forward Guidance Puzzle," Federal Reserve Bank of New York working paper.

[11] Dreze, J., 1975, "Existence of an Exchange Equilibrium under Price Rigidities," *International Economic Review* 16, 301-320.

[12] Farhi, E., and Werning, I, 2017, "Monetary Policy, Bounded Rationality and Incomplete Markets, NBER working paper 23281.

[13] Gabaix, X., 2018, "A Behavioral New Keynesian Model," Harvard University working paper.

[14] Gali, J., 2015, *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Model and its Applications, 2nd Edition,* Princeton, NJ: Princeton University Press.

[15] Garcia-Schmidt, M. and Woodford, M., 2015, "Are Low Interest Rates Deflationary? A Paradox of Perfect Foresight Analysis," NBER working paper 21614.

[16] Jackson, M., 1992, "Implementation in Undominated Strategies: A Look at Bounded Mechanisms," *Review of Economic Studies* 59, 757-775.

[17] Kaplan, G., Moll, B., and Violante, G., 2018, "Monetary Policy According to HANK," *American Economic Review* 108, 697-743.

[18] Kocherlakota, N., 1992, "Bubbles and Constraints on Debt Accumulation," *Journal of Economic Theory* 57, 245-56.

[19] Kocherlakota, N., and Phelan, C., Fall 1999, "Explaining the Fiscal Theory of the Price Level," *Federal Reserve Bank of Minneapolis Quarterly Review*, 14-23.

[20] Leeper, E., 1991, "Equilibria Under 'Active' and 'Passive' Monetary and Fiscal Policies," *Journal of Monetary Economics* 27, 129-147.

[21] McKay, A., Nakamura, E., and Steinsson, J., "The Power of Forward Guidance Revisited," *American Economic Review* 106, 3133-58.

[22] Mongey, S., 2018, "Market Structure and Monetary Non-Neutrality," working paper, The University of Chicago.

[23] Obstfeld, M., and Rogoff, K., "Speculative Hyperinflations in Maximizing Models: Can We Rule Them Out?" *Journal of Political Economy* 91, 675-87.

[24] Sargent, T., and Wallace, N., 1975, "'Rational Expectations', the Optimal Monetary Instrument, and the Optimal Money Supply Rule," *Journal of Political Economy* 83, 241-54.

[25] Werning, I., 2015, "Incomplete Markets and Aggregate Demand," NBER working paper 21448.

[26] Woodford, M., 1995, "Price Level Determinacy without Control of a Monetary Aggregate," *Carnegie-Rochester Conference Series on Public Policy* 43, 1-46.

Appendix A

In this appendix, I generalize Section 2.2 to settings with sticky prices and storage. As there, I show that, with bounded prices, there is a sequence of equilibria in which consumption converges to zero.

Sticky Prices

In this section, I assume that (in the setup of Section 2.2) a fraction θ of firms must set their prices equal to P^{fix}. The other firms choose their prices from the interval Λ. As in the body of the paper, I will consider two possibilities: $\Lambda = (0, \infty)$ and $\Lambda = [P_{min}, P_{max}]$. I show that:

- equilibrium consumption is necessarily bounded from below by a positive number if the flex-price firms choose from an unbounded interval.
- there is a sequence of equilibria in which consumption converges to zero if the flex-price firms choose from a bounded and closed interval.

The model without bounds misses a key dynamic feedback from period 2 inflation (expectations) onto period 1 outcomes.

Suppose first that **the flex-price firms can choose any positive price**, so that $\Lambda = (0, \infty)$. Then, we can show that an equilibrium is a specification of $(C_1^{fix}, C_1^{flex}, C_1^*, N_1^*, P_1^*, \Pi_2^*, P_1^{flex}, W^*)$ such that:

$$
\begin{aligned}
P_1^* &= (\theta(P^{fix})^\eta + (1-\theta)(P_1^{flex})^\eta)^{1/\eta} \\
P_1^{flex} &= W^*(1-1/\eta)^{-1} \\
C_1^* &= (\theta(C_1^{fix})^{1-1/\eta} + (1-\theta)(C_1^{flex})^{1-1/\eta})^{\frac{\eta}{\eta-1}} \\
C_1^{fix}(P^{fix})^\eta &= C_1^{flex}(P_1^{flex})^\eta \\
W^* &= v'(N_1^*)P_1^*/u'(C_1^*) \\
N_1^* &= (\theta C_1^{fix} + (1-\theta)C_1^{flex}) \\
u'(C_1^*) &= \beta R u'(Y)/\Pi_2^*
\end{aligned}
$$

There is a relative price distortion in this economy caused by the restriction that some firms cannot change their prices. But there is a limit to the damage that this distortion can cause: the worst that can happen is that the fixed-price firms do not produce at all. Hence, equilibrium consumption is bounded from below[16] by C_{LB} which satisfies:

$$u'(C_{LB})(1-\theta)^{1/(\eta-1)}(1-1/\eta) = v'(C_{LB}(1-\theta)^{1/(1-\eta)}).$$

Now suppose that the **flex-price firms choose their prices from a closed and bounded interval** $[P_{min}, P_{max}]$. Then, an equilibrium is a specification of $(C_1^{fix}, C_1^{flex}, C_1^*, N_1^*, P_1^*, \Pi_2^*, P_1^{flex},$ such that:

[16]Many applications of sticky price models use log-linear approximations. But such an approximation is necessarily too crude to reveal that consumption is bounded from below in this fashion. I thank Ivan Werning for pointing this out to me in a private communication.

$$P_1^* = (\theta(P_1^{fix})^\eta + (1-\theta)(P_1^{flex})^\eta)^{1/\eta}$$

$$P_1^{flex} = max(P_{min}, min(P_{max}, W^*(1-1/\eta)^{-1}))$$

$$C_1^* = (\theta(C_1^{fix})^{1-1/\eta} + (1-\theta)(C_1^{flex})^{1-1/\eta})^{\frac{\eta}{\eta-1}}$$

$$C_1^{fix}(P^{fix})^\eta = C_1^{flex}(P_1^{flex})^\eta$$

$$W^* = v'(N_1^*)P_1^*/u'(C_1^*)$$

$$N_1^* = (\theta C_1^{fix} + (1-\theta)C_1^{flex})$$

$$u'(C_1^*) = \beta R u'(Y)/\Pi_2^*$$

Pick any Π_2^* that is near zero. Then, I conjecture that there is an equilibrium in which $P_1^* = P_{min}$. In that equilibrium:

$$u'(C_1^*) = \beta R u'(Y)/\Pi_2^*,$$

where:

$$\begin{aligned}
C_1^{flex} &= C_1^*(\theta\alpha^{1-1/\eta} + (1-\theta))^{\frac{\eta}{1-\eta}} \\
\alpha &= (P_{fix}/P_{min})^{-\eta} \\
N_1^* &= (\theta\alpha C_1^{flex} + (1-\theta)C_1^{flex}) \\
P_1^* &= (\theta(P^{fix})^\eta + (1-\theta)(P_{min})^\eta)^{1/\eta} \\
W^* &= v'(N_1^*)P_1^*/u'(C_1^*) < (1-1/\eta)P_{min}
\end{aligned}$$

By taking a sequence of Π_2^*'s that converge to zero, we can find a sequence of equilibria in which C_1^* also converge to zero.

Thus, if there is a lower bound on firm prices, there is a sequence of equilibria, indexed by period 2 inflation, in which consumption converges to zero. Note that this conclusion is independent of the lower bound P_{min} and is also independent of θ.

Storage

I now return to the assumption that all firms can choose their prices from the set Λ (so there are no sticky price firms). However, I introduce a new technology: I suppose agents can store x units of consumption in period 1 to generate $(1+\phi)x$ units of consumption in period 2, for any $x \geq 0$. I assume that ϕ is sufficiently large that:

$$u'(Y^{real}) < \beta(1+\phi)u'(Y)$$

where, as in Sections 3-4, Y^{real} is defined to be the solution to:

$$u'(Y^{real})(1-1/\eta) = v'(Y^{real}).$$

This assumption ensures that storage would be positive in the absence of money.

Suppose first that **the firms can choose any positive price**, so that $\Lambda = (0, \infty)$. Then, the unique equilibrium allocation of consumption C_1^* and storage S^* satisfies:

$$\begin{aligned} u'(C_1^*) &= \beta(1+\phi)u'(Y+(1+\phi)S^*) \\ u'(C_1^*)(1-1/\eta) &= v'(C_1^*+S^*) \end{aligned}$$

and period 2 inflation Π^* must satisfy:

$$(1+\phi) = R/\Pi^*$$

Note that this equilibrium depends on agents in period 2 co-ordinating on the price level:

$$\Pi^* = R/(1+\phi).$$

Now suppose instead that $\Lambda = [P_{min}, P_{max}]$, so that **firms choose their prices from a compact interval**. Pick Π_2^* sufficiently small so that money dominates storage as a store of value:

$$1/\Pi_2^* > (1 + \phi).$$

Then there is an equilibrium in which:

$$\begin{aligned} u'(C_1^*) &= \beta u'(Y)/\Pi_2^* \\ S^* &= 0 \\ u'(C_{1,}^*) \quad (1 - 1/\eta) &= v'(C_1^*) \end{aligned}$$

By considering a sequence of Π_2^* that converge to zero, we can construct a corresponding sequence of equilibria in which C_1^* and S^* both converge to zero. Again, this conclusion is independent of P_{min}.

Appendix B

In this appendix, I gather the remaining proofs.

Proof of Proposition 1

The necessity of the first order conditions is straightforward. The necessity of the transversality condition follows from a standard argument. Suppose:

$$lim_{t\to\infty}\beta^t u'(C_t^*)\bar{M}_t/P_t^* = L > 0.$$

I claim that it is possible to find a budget-feasible perturbation that makes the household better off. Thus, given ε in $(0, L)$, there exists T such that:

$$\beta^t u'(C_t^*)\bar{M}_t/P_t^* > \varepsilon$$

for all $t \geq T$. Consider a perturbation whereby the household increases consumption at date t by $\beta^{-t}\varepsilon/u'(C_t^*)$, lowers M_t by $\beta^{-t}\varepsilon u'(C_t^*)^{-1}/P_t^*$, and lower $M_{t+s}, s \geq 1$, by:

$$\begin{aligned}
&(\varepsilon\beta^{-t}u'(C_t^*)^{-1}/P_t^*)\prod_{\tau=1}^{s} R_{t+\tau}^* \\
&= (\varepsilon\beta^{-t}u'(C_t^*)^{-1}/P_t^*)\beta^{-s}(u'(C_t^*)/u'(C_{t+s}^*))(P_t^*/P_{t+s}^*) \\
&= \frac{\varepsilon\beta^{-t-s}}{P_{t+s}^* u'(C_{t+s}^*)} \\
&< \bar{M}_{t+s}
\end{aligned}$$

This perturbation is budget-feasible (because the household's moneyholdings remain positive in all future periods.)

The sufficiency of the price-setting first order condition as a solution to the firm's problem is obvious. The sufficiency of the other conditions for household optimality is by contradiction. Suppose (C', N', M') is budget-feasible and dominates $(C^*, N^*, \bar{M})$. That means:

$$
\begin{aligned}
0 &< \sum_{t=1}^{\infty} \beta^{t-1}(u(C_t') - v(N_t')) - \sum_{t=1}^{\infty} \beta^{t-1}(u(C_t^*) - v(N_t^*)) \\
&= \sum_{t=1}^{T} \beta^{t-1}(u(C_t') - v(N_t')) - (u(C_t^*) - v(N_t^*)) \\
&+ \beta^T \sum_{t=T+1}^{\infty} \beta^{t-1-T}(u(C_t') - v(N_t')) - (u(C_t^*) - v(N_t^*))
\end{aligned}
$$

If we take limits with respect to T and since (u, v) are bounded from above and below, we find that:

$$
0 < lim_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1}(u(C_t') - v(N_t')) - (u(C_t^*) - v(N_t^*))
$$

Next, we can use the subgradient inequality for concave functions:

$$
\begin{aligned}
0 &< lim_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1}(u'(C_t^*)(C_t' - C_t^*) - v'(N_t^*)(N_t' - N_t^*)) \\
&= lim_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1}(u'(C_t^*)((C_t' - C_t^*) - W_t^*(N_t' - N_t^*)/P_t^*) \\
&= lim_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1}(u'(C_t^*))((M_{t-1}' - \bar{M}_{t-1})R_t^*/P_t^* - (M_t' - \bar{M}_t)/P_t^*) \\
&= lim_{T\to\infty} \beta^{T-1}(u'(C_T^*))(\bar{M}_T - M_T')/P_T^*) \\
&\leq lim_{T\to\infty} \beta^{T-1} u'(C_T^*)\bar{M}_T/P_T^* \\
&= 0.
\end{aligned}
$$

where the penultimate step comes from the non-negativity of M'. This contradiction proves the proposition.

Proof of Proposition 5

There are two distinct cases based on the magnitude of the parameter γ, defined as:

$$\gamma = \beta^{-1}\pi_{min}/\hat{R}(\pi_{min}). \tag{12}$$

Suppose first that $\gamma \geq 1$ (so that the average real return to money is no higher than $1/\beta$ when inflation is at its lowest level). Define $(\lambda_k)_{k=1}^{\infty}$ to be any strictly increasing sequence that converges to infinity with initial $\lambda_1 > 1$. Define $(C_t^{k*})_{t=1}^{\infty}$ via the Euler equation:

$$u'(C_{t+1}^{k*}) = \lambda_k \gamma^{t-1} u'(Y^{real}), t = 1, 2, ...$$

We can readily verify that for all (k, t):

$$u'(C_{t+1}^{k*}) > u'(Y^{real}).$$

Define $\pi_t^{k*} = \pi_{min}$ and $w_t^{k*} = v'(C_t^{k*})/u'(C_t^{k*})$ for all (k, t). Then, we can verify that using the conditions in Proposition 1 that $(C^{k*}, w^{k*}, \pi^{k*})$ is part of an equilibrium. Note that for any t:

$$lim_{k\to\infty} u'(C_t^{k*}) \geq \gamma^{t-1} u'(Y^{real}) lim_{k\to\infty} \lambda_k = \infty$$

and so $lim_{k\to\infty} C_t^{k*} = 0 = lim_{k\to\infty} W^{k*}$.

The second case is that, as defined in (12), $\gamma < 1$ (intuitively, the average long-run real return to money is higher than $1/\beta$ when inflation equals π_{min}). In that case, let $\hat{\pi}$ satisfy:

$$\pi_{min} < \hat{\pi} < \pi^{TAR}$$

$$\beta \hat{R}(\pi_{min}) > \hat{\pi}$$

(There is such a value for $\hat{\pi}$ because R is continuous.) Pick any horizon $k > 1$. Given k, define an inflation sequence π^{k*} recursively as:

$$\pi_{t+1}^{k*} = \beta \hat{R}(\pi_t^{k*}), t \geq k$$
$$\pi_k^{k*} = \hat{\pi}$$
$$\pi_t^{k*} = \pi_{min}, t < k$$

and define a consumption sequence C^{k*} so that:

$$C_t^{k*} = Y^{real}, t \geq k$$
$$u'(C_t^{k*}) = \gamma^{t-k-1} \frac{\beta \hat{R}(\pi_{min}) u'(Y^{real})}{\pi_k^{k*}}, 1 \leq t < k \tag{13}$$

Since $\gamma < 1$, $u'(C_t^{k*}) > u'(Y^{real})$ for $t < k$.

We know that:

$$\pi_{k+1}^{k*} = \beta \hat{R}(\hat{\pi}) \geq \beta \hat{R}(\pi_{min}) > \hat{\pi} = \pi_k^{k*}$$

We know too that, since $\pi^{TAR} > \pi_k^{k*}$, $\pi^{TAR} \geq \pi_{k+1}^{k*}$. Since $\hat{R}$ is weakly increasing, induction implies that:

$$\pi^{TAR} \geq \pi_{t+1}^{k*} \geq \pi_t^{k*} \geq \pi_{min}$$

for all $t \geq k$, and that the sequence $(\pi_t^{k*})_{t \geq k}$ converges to the smallest fixed point of $\beta \hat{R}$ that is larger than π_{min}. We can then verify using Proposition 1 that (C^{k*}, π^{k*}) is part of an equilibrium.

It is clear that $lim_{k \to \infty} \pi_t^{k*} = \pi_{min}$ for all t. And since $\gamma < 1, lim_{k \to \infty} u'(C_t^{k*}) = \infty$ for all t. It follows too that $lim_{k \to \infty} W^{k*} = 0$.

Proof of Proposition 6

There are two cases.

Case 1: $\beta \hat{R}(\pi_{min}) \leq \pi_{min}$.

The proof for this case is by contradiction. Suppose C^* is part of an equilibrium and $C_t^* < Y^{real}$. Then:

$$\begin{aligned} u'(C_{t+1}^*) &= \frac{\beta^{-1}u'(C_t^*)}{\hat{R}(\pi_{min})}\pi_{t+1}^* \\ &= \frac{\beta^{-1}u'(C_t^*)}{\hat{R}(\pi_{min})/\pi_{min}}\frac{\pi_{t+1}^*}{\pi_{min}} \\ &\geq u'(C_t^*). \end{aligned}$$

Hence, by induction, $\pi_{t+s}^* = \pi_{min}$ for all $s \geq 0$. The households' transversality condition requires that:

$$\begin{aligned} 0 = u'(C_t^*) lim_{s\to\infty} \frac{\beta^s u(C_{t+s}^*)\bar{M}_{t+s}}{u'(C_t^*)(\pi_{min})^s P_t^*} \\ = u'(C_t^*) lim_{s\to\infty} \frac{M_0(t+s)^{-1}\hat{R}(\pi^{TAR})^s}{\hat{R}(\pi_{min})^s} \\ = \infty \end{aligned}$$

which is a contradiction.

Case 2: $\beta\hat{R}(\pi_{min})/\pi_{min} > 1$.

Define $\hat{\pi} \in (\pi_{min}, \pi^{TAR} - \epsilon)$ so that it satisfies:

$$\beta\hat{R}(\hat{\pi}) = \hat{\pi}$$

$$\beta\hat{R}(\pi) > \pi$$

for all π in $[\pi_{min}, \hat{\pi})$. We know such a $\hat{\pi}$ exists because $\beta\hat{R}(\pi^{TAR} - \varepsilon) < \pi^{TAR} - \varepsilon$ and $\beta\hat{R}(\pi_{min}) > \pi_{min}$.

Suppose $C_t^* < Y^{real}$ at some date t. I show first, by contradiction, that there is some

$s \geq 0$ such that $C^*_{t+s+1} \geq Y^{real}$. Suppose not. Then for all $s \geq 0$,

$$u'(C^*_{t+s+1}) = [\beta^{-1}\hat{R}(\pi_{min})^{-1}\pi_{min}]^{s+1}u'(C^*_t)$$

But this implies that $u'(C^*_{t+s+1})$ is lower than $u'(Y^{real})$ for s sufficiently large, which is the desired contradiction.

Hence, there is some $s \geq 0$ such that $C^*_{t+s+1} \geq Y^{real}$ and $C^*_{t+s} < Y^{real}$. It follows that:

$$\begin{aligned}
\pi^*_{t+s+1} &= \beta\hat{R}(\pi_{min})u'(C^*_{t+s+1})/u'(C^*_{t+s}) \\
&< \beta\hat{R}(\pi_{min})u'(C^*_{t+s+1})/u'(Y^{real}) \\
&\leq \beta\hat{R}(\hat{\pi}) \\
&= \hat{\pi}.
\end{aligned}$$

Since $\hat{\pi} < \pi_{max}$, it follows that $C^*_{t+s+1} = Y^{real}$ and:

$$\pi^*_{t+s+1} < \beta\hat{R}(\pi_{min}) \leq \hat{\pi}.$$

Since $\beta\hat{R}(\pi^*_{t+s+1}) > \pi^*_{t+s+1}$, we can conclude that:

$$\begin{aligned}
\frac{u'(C^*_{t+s+2})}{\pi^*_{t+s+2}} &= [\beta\frac{\hat{R}(\pi^*_{t+s+1})}{\pi^*_{t+s+1}}]^{-1}\frac{u'(Y^{real})}{\pi^*_{t+s+1}} \\
&< u'(Y^{real})/\pi^*_{t+s+1}.
\end{aligned}$$

and so $C^*_{t+s+2} \geq Y^{real}$. Hence:

$$\begin{aligned}\pi^*_{t+s+2} &= \beta\hat{R}(\pi^*_{t+s+1})u'(C^*_{t+s+2})/u'(Y^{real}) \\ &\leq \beta\hat{R}(\pi^*_{t+s+1}) \\ &< \beta\hat{R}(\hat{\pi}) \\ &= \hat{\pi} < \pi_{max}.\end{aligned}$$

It follows that $C^*_{t+s+2} = Y^{real}$ and so:

$$\begin{aligned}\pi^*_{t+s+2} &= \beta\hat{R}(\pi^*_{t+s+1}) \\ &> \pi^*_{t+s+1.}.\end{aligned}$$

By induction, we can conclude that for all $r \geq 1$:

$$\begin{aligned}C^*_{t+s+r} &= Y^{real} \\ \pi^*_{t+s+r+1} &= \beta\hat{R}(\pi^*_{t+s+r}),\end{aligned}$$

where $\pi^*_{t+s+1} = \beta\hat{R}(\pi_{min})$. The sequence $(\pi^*_{t+s+r})^\infty_{r=1}$ is strictly increasing, and converges to $\hat{\pi}$ (the smallest fixed point of $\beta\hat{R}$ which is greater than π_{min}).

To be an equilibrium, the households' transversality condition must be satisfied:

$$\begin{aligned}0 &= lim_{T\to\infty}\bar{M}_{t+T}/\prod_{r=1}^{T}\hat{R}(\pi^*_{t+r}) \\ &\geq lim_{T\to\infty}\bar{M}_{t+T}/(\hat{R}(\hat{\pi}))^T \\ &= \bar{M}_t lim_{T\to\infty}(t/(t+T))(\frac{\hat{R}(\pi^{TAR})}{\hat{R}(\hat{\pi})})^t \\ &= \infty.\end{aligned}$$

But this is a contradiction: the nominal liabilities are growing too fast to be consistent with an equilibrium in which inflation is bounded from above by $\hat{\pi}$. It follows that there cannot be any $C_t^* < Y^{real}$.

Proof of Proposition 7

We proceed by induction. Suppose that $C_t^* > Y^{real}$ for some t. Then $\pi_t^* = \pi_{max}$ and:

$$\begin{aligned} u'(C_{t+1}^*) &= \beta^{-1} u'(C_t^*) \pi_{t+1}^* / \hat{R}(\pi_{max}) \\ &= \beta^{-1} u'(C_t^*)(\pi_{t+1}^*/\pi_{max}) \pi_{max} / \hat{R}(\pi_{max}) \\ &\leq u'(C_t^*)[\frac{\beta^{-1}\pi_{max}}{\hat{R}(\pi_{max})}] \end{aligned}$$

By induction, it follows that:

$$u'(C_{t+s}^*) \leq u'(C_t^*)[\frac{\beta^{-1}\pi_{max}}{\hat{R}(\pi_{max})}]^s, s \geq 1.$$

Given the Inada condition on u', this implies that

$$lim_{s\to\infty} C_{t+s}^* = \infty.$$

But this violates the requirement that:

$$C_t^* - C_t^* v'(C_t^*)/u'(C_t^*) \geq -\Upsilon.$$